SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 1st December 2017

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

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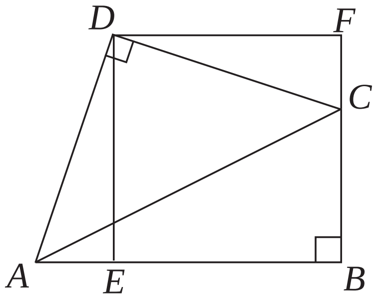
1. **132** Each circle immediately adjacent to the initial '4' must contain the same integer, x say, in order for the sum of those numbers at the end of each line segment to be the same. Those circles immediately adjacent to those with the integer x must contain the integer 4 to preserve the sum of those numbers at the end of each line segment. Continuation of this pattern throughout the network eventually yields that the circle marked with a '7' must contain the integer x. Therefore $x = 7$. The completed network contains twelve '4's and twelve '7's with a total of $12 \times 4 + 12 \times 7 = 132$.

2. **231** At the start of the race, the runners are in order 123. By the end of the race Primus and Tertius have exchanged places 11 times, so Tertius ends the race ahead of Primus. Also, since Secundus and Tertius have exchanged places 10 times, Secundus ends ahead of Tertius. So the result of the race is 231.

3. **221** In ascending order the list is 104, 113, 122, 131, 140, 203, 212, 221, 230, 302, 311, 320, 401, 410, 500. The median of this list is 221.

4. **216** Each of the squares has a side-length of 18 cm. Therefore each of the triangles has two sides of length 18 cm. Hence the triangles are isosceles. Let the angle contained by the two 18 cm sides of these triangle be $x°$. The interior angles of a square and a regular hexagon are 90° and 120° respectively. By considering angles at a point we have $x + 90 + 90 + 120 = 360$. Therefore $x = 60$ and the triangles are equilateral. All twelve outer edges of the figure are 18 cm in length. Therefore $P = 12 \times 18 = 216$.

5. **625**



Draw a line through D that is parallel to AB. Let F be the intersection of that line with BC extended, as shown in the diagram.

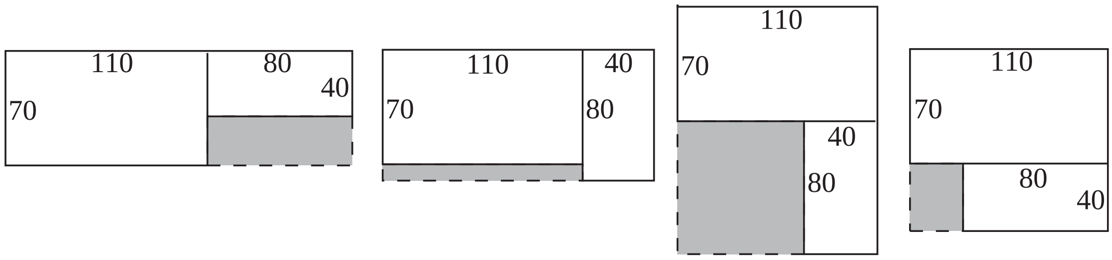
Now	$\angle EDC + \angle CDF = \angle EDC + \angle ADE = 90°$
Therefore	$\angle CDF = \angle ADE = 90° - \angle ECD$
Also	$\angle DFC = \angle DEA = 90°$

Therefore triangles ADE and CDF are similar because they have the same set of angles. Because $AD = CD$ they must also be congruent.

Therefore the area of quadrilateral $ABCD$ is equal to the area of rectangle $FDEB$. By the congruent triangles ADE and CDF we know that $DF = DE = 25$. Therefore the required area is $25 \times 25 = 625$.

6. **373** Winnie begins with 2017 integers. There are $2016 \div 3 = 672$ multiples of three which are erased. This leaves $2017 - 672 = 1345$ integers. There are $2016 \div 6 = 336$ multiples of six which are reinstated. This leaves $1345 + 336 = 1681$ integers. The only multiples of twenty-seven that then remain in the list are those that are multiples of six. Winnie therefore erases all the multiples of fifty-four. There are $1998 \div 54 = 37$ multiples of fifty-four which are erased. This leaves $1681 - 37 = 1644$ integers. Of the 2017 integers she began with $2017 - 1644 = 373$ are now missing.

7. **300**



There are essentially four different configurations as shown in the diagram. The perimeters of the third rectangle in these configurations are 220, 240, 300 and 140 respectively. Therefore the maximum possible perimeter of the third rectangle is 300.

8. **276** Let the angle that each hand makes with the vertical be x degrees and let the current time be s seconds after midday.

In one complete hour the hour hand will turn 30°. There are $60 \times 60 = 3600$ seconds in an hour so it takes $3600 \div 30 = 120$ seconds for the hour hand to turn one degree. Therefore $s = 120x$.

In one complete hour the minute hand will turn 360°. There are 3600 seconds in an hour so it takes $3600 \div 360 = 10$ seconds for the minute hand to turn 1°. But the minute hand has turned clockwise through an angle of $(360 - x)°$. Therefore $s = 10(360 - x)$. Equating the two expressions we have obtained for s we obtain the equation $120x = 10(360 - x)$. The solution to this equation is $x = \frac{360}{13}$. Therefore the number of seconds elapsed since midday is $120 \times \frac{360}{13} = \frac{43200}{13} = 3323\frac{1}{13}$. The number of whole seconds remaining is $3600 - 3324 = 276$.

9. **013** Robin could score a total of zero either by missing the target with all three arrows or if any two of his arrows hit adjacent regions.

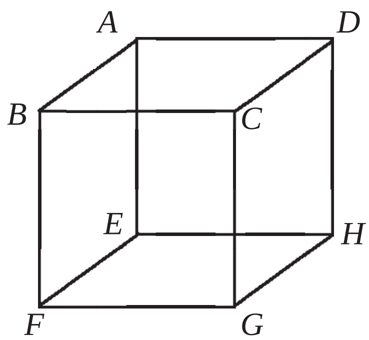
Robin could score totals of 3, 9, 15, 21, 27 or 33 if all three of his arrows hit regions 1, 3, 5, 7, 9 or 11 respectively.

The only scores Robin can obtain from his three arrows hitting non-adjacent regions are $1 + 1 + 5 = 7$, $1 + 5 + 5 = 11$, $3 + 3 + 7 = 13$, $3 + 7 + 7 = 17$, $9 + 9 + 11 = 29$, $9 + 11 + 11 = 31$.

Robin's set of possible score is therefore {0, 3, 7, 9, 11, 13, 15, 17, 21, 27, 29, 31, 33}.

Hence Robin can obtain 13 different possible scores.

10. **004** Suppose that there is a truther at A. There must be two liars and one truther adjacent to A. Let us suppose, without loss of generality, that B is a truther and D and E are liars. Since B is a truther and is adjacent to A, then C and F are liars. This shows that there cannot be more than 4 truthers.



If we now suppose that G and H are both truthers, then each of the Bunchkins' statements fits the conditions. So 4 is the maximum possible number of Bunchkins.

11. 714 The difference between any two terms is either 4 or is a multiple of 4. So the term-to-term difference in the progression must be a divisor of 4. Since all the terms of the progression are integers the only feasible differences are 1, 2 and 4.

If the term-to-term difference is 1 then the one-hundredth term will be $7 + 99 \times 1 = 106$.

If the term-to-term difference is 2 then the one-hundredth term will be $7 + 99 \times 2 = 205$.

If the term-to-term difference is 4 then the one-hundredth term will be $7 + 99 \times 4 = 403$.

The sum of these numbers is 714.

12. 121 Take each shaded semi-annulus that is below the line and reflect it in the line then move it one centimetre to the left. A shaded semicircle of diameter 11cm is obtained. Therefore the whole shaded area is $\frac{121}{8}\pi$. Hence the value of k is 121.

13. 002 The expression may be simplified to $\dfrac{k \cdot a \cdot n \cdot r \cdot o \cdot o}{m \cdot e}$.

The smallest possible numerator is $5 \times 4 \times 3 \times 2 \times 1 \times 1 = 120$.

The largest possible denominator is $9 \times 8 = 72$.

The smallest possible value of the expression, whether integer or not, is therefore $\dfrac{120}{72} = 1\dfrac{2}{3}$.

We may obtain a value of 2 for the expression via $\dfrac{6 \times 4 \times 3 \times 2 \times 1}{9 \times 8}$.

Therefore the smallest possible integer value of the expression is 2.

14. 027 There are six subsets consisting of one element:

{1}, {2}, {3}, {4}, {5}, {6}.

There are eleven subsets consisting of two elements:

{1, 2}, {1,3}, {1,4}, {1, 5}, {1, 6}; {2, 3}, {2, 5}; {3, 4}, {3, 5}: {4, 5}; {5, 6}.

There are eight subsets consisting of three elements:

{1, 2, 3}, {1, 2, 5}; {1, 3, 4}, {1, 3, 5}; {1, 4, 5}; {1, 5, 6}; {2, 3, 5}; {3, 4, 5}.

There are two subsets consisting of four elements:

{1, 2, 3, 5}; {1, 3, 4, 5}.

Hence the answer is $6 + 11 + 8 + 2 = 27$.

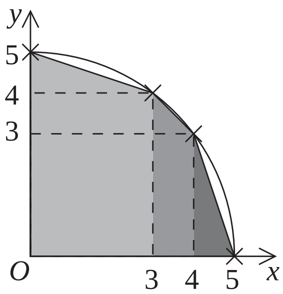
15. 829 The possible answers for 1 across are 16, 25, 36, 49, 64 and 81.

The possible answers for 2 down are 124, 215, 342, 511, 728, 999.

By considering the last digit of 1 across (which must be the same as the first digit of 2 down) we see that the only possible pairs of answers for 1 across and 2 down are (25, 511), (49, 999) and (81, 124). The pair (49, 999) would leave no possible answer for 5 across, so may be disregarded. The pair (25, 511) gives an answer of 16 for 5 across and thence 36 for 4 down and 25 for 1 down. However these answers contradict the clue for 4 down so this case may be disregarded. (81, 124) gives an answer of 49 for 5 across and hence 99 for 4 down and 88 for 1 down. These answers satisfy all the conditions in the clues and therefore the answer to this Kangaroo question is 829.

16. 074 The curve $x^2 + y^2 = 25$ is a circle of radius 5 centred at the origin.

The polygon P has vertices at coordinates (0,5), (3,4), (4,3), (5,0), (4,–3), (3,–4), (0,–5), (–3, –4), (–4, –3), (–5, 0), (–4, 3), (–3, 4).



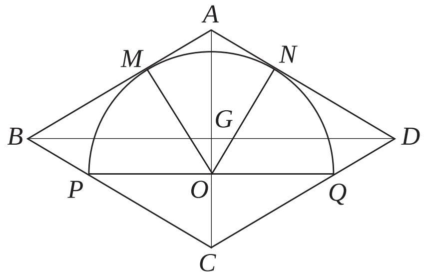
We may find the area of that part of P in the upper-right quadrant by splitting it into two trapezia and a triangle as shown in the diagram. This area is

$$\left(\tfrac{1}{2}(5 + 4) \times 3\right) + \left(\tfrac{1}{2}(4 + 3) \times 1\right) + \left(\tfrac{1}{2} \times 3 \times 1\right) = \tfrac{37}{2}.$$

Therefore the area of P is $4 \times \frac{37}{2} = 74$.

17. 447 Three-digit squares beginning with 2, 4, 5, 6 or 8 may be disregarded since on reversal these will be divisible by two or five. The residual three-digit squares are 100, 121, 144, 169, 196, 324, 361, 729, 784, 900 and 961. We may disregard 144, 324, 729 and 961 since these are divisible by three and will remain so on reversal. We may also disregard 169, 961, 100 and 121 since these each form a square on reversal. This leaves 196, 361 and 784 which on reversal form 163, 487 and 691. Each of these numbers is prime and they have a mean of $\frac{1}{3}(163 + 487 + 691) = 447$.

18. 603 Let O be the centre of the semicircle and let M and N be the feet of the perpendiculars drawn from O to AB and AD respectively. Let G be the intersection of the diagonals of the rhombus.



$PO = 10$ and $\angle OPC = 30°$. So $OC = 10\tan 30° = \dfrac{10}{\sqrt{3}}$.

$MO = 10$ and $\angle OAM = 60°$. So $AO = \dfrac{10}{\sin 60°} = \dfrac{20}{\sqrt{3}}$.

Therefore $AC = \dfrac{10}{\sqrt{3}} + \dfrac{20}{\sqrt{3}} = \dfrac{30}{\sqrt{3}} = 10\sqrt{3}$.

Hence $AG = 5\sqrt{3}$ and $\angle GBA = 30°$. So $BG = \dfrac{5\sqrt{3}}{\tan 30°} = 15$.

Therefore the area of triangle BGA is $\frac{1}{2} \times 15 \times 5\sqrt{3} = \frac{75}{2}\sqrt{3}$.

So the area of the rhombus is $4 \times \frac{75}{2}\sqrt{3} = 150\sqrt{3}$.

Therefore $a = 150$ and $b = 3$, so $ab + a + b = 450 + 150 + 3 = 603$.

19. 343

$$F_1(x) = x;\ F_2(x) = \frac{1}{1-x};\ F_3(x) = \frac{1}{1-\frac{1}{1-x}} = 1 - \frac{1}{x};\ F_4(x) = \frac{1}{1-\left(1-\frac{1}{x}\right)} = x.$$

Hence we have, $F_1(x) = F_4(x) = F_7(x) = \ldots = F_{3k-2}(x)$.

Therefore we are required to solve $F_{3k-2}(3k-2) = 3k - 2$ where $3k - 2$ is a three-digit cube (given). The cubes are 125, 216, 343, 512, 729. The only one of the format $3k - 2$ for some positive integer k is 343.

20. 077 It is clear that each of a, b and c must have prime factors including 2 and 3 and, since we are seeking a minimal number of factors of abc, these must be the only prime factors.

Let $a = 2^p3^q$, $b = 2^r3^s$ and $c = 2^v3^w$ where p, q, r, s, v and w are positive integers or zero. Since $a^2 = 2b^3 = 3c^5$ we have $2^{2p}3^{2q} = 2^{3r+1}3^{3s} = 2^{5v}3^{5w+1}$.

Considering indices of 2 we have $2p = 3r + 1 = 5v$.

The smallest values of (p, r, v) which satisfy this equation are (5, 3, 2).

Considering indices of 3 we have $2q = 3s = 5w + 1$.

The smallest values of (q, s, w) which satisfy this equation are (3, 2, 1).

Using these values, $abc = \left(2^5 3^3\right) \times \left(2^3 3^2\right) \times \left(2^2 3^1\right) = 2^{10}3^6$.

Any factors of abc will be of the form 2^y3^z where $y \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $z \in \{0, 1, 2, 3, 4, 5, 6\}$.

There are 11 possibilities for y and 7 possibilities for z. Hence abc has $11 \times 7 = 77$ factors.